

A NOTE ON A THEOREM OF R DUFFIN(U) WISCONSIN

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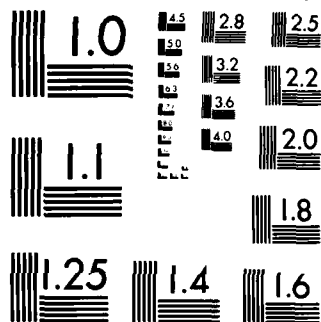
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A NOTE ON A THEOREM OF R. DUFFIN

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ABSTRACT

In an earlier paper, Duffin studied a model for the transverse vibrations of a string of n beads. This note looks at the same question in a more general setting. It establishes the existence of special solutions corresponding to high frequency oscillations which are "positive" in the coordinate system chosen as well as other "non-positive" solutions.

AMS(MOS) Subject Classifications: 34B15, 47H15

Key Words: transverse vibrations of a string, continua of positive solutions, bifurcation.

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

In an earlier paper, Duffin modelled the transverse vibration of a string of n beads by the system of ordinary differential equations

$$m_i x_i'' = f_i(x_{i-1} + x_i) + f_{i+1}(x_i + x_{i+1}), \quad 1 \leq i \leq n$$

where the positive coordinate directions are as indicated on page 1. He established the existence of families of solutions all components of which are initially positive (in this coordinate system) with zero velocity, and after a time interval T all components have zero amplitude. We prove the existence of such highly oscillatory solutions in a more general setting and also study the existence of nonpositive solutions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NOTE ON A THEOREM OF R. DUFFIN

Paul H. Rabinowitz

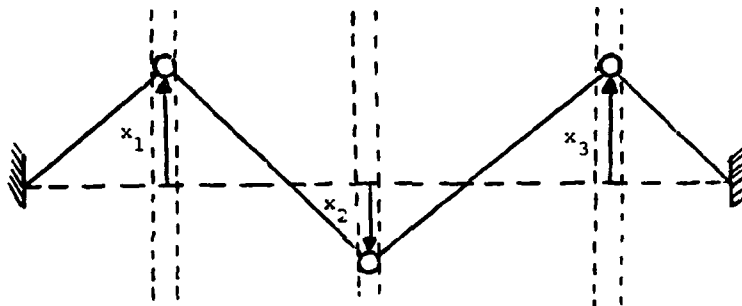
§1. Introduction

In [1], R. Duffin studied the transverse vibrations of a string of n beads stretched between two fixed end points. Assuming that longitudinal motion was constrained by channels and that tension only depends on the extension of the string segment, the motion was modelled by the system of equations

$$(1.1) \quad m_i x_i'' = f_i(x_{i-1} + x_i) + f_{i+1}(x_i + x_{i+1}), \quad 1 \leq i \leq n$$

where the positive coordinate directions are as indicated in the figure below,

$x_0 \equiv 0 \equiv x_{n+1}$, and $m_i > 0$ is the mass of the i^{th} bead. Let P denote the cone of



positive vectors in \mathbb{R}^n , i.e. $P = \{x = (x_1, \dots, x_n) \mid x_i > 0, 1 \leq i \leq n\}$. Thus in the above coordinate system, if at time t , the beads lie in P , $x_1(t)$ lies above the line, ℓ , joining the endpoints of the string, x_2 lies below ℓ , etc. In particular adjacent points have displacements of opposite sign relative to ℓ . Duffin used an elegant argument based on the Brouwer Fixed Point Theorem to prove

Theorem 1.2. Suppose the functions f_i are Lipschitz continuous and there is a constant $h > 0$ such that

$$(1.3) \quad -f_i(s) > hs, \quad 1 \leq i \leq n+1$$

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for all $s > 0$. Let $S \subset \mathbb{R}^n$ be a surface bounding a starshaped neighborhood of 0. Then there is an $x^* \in S \cap P$ and $T^* > 0$ such that the initial value problem for (1.1) with $x(0) = x^*$ and $x'(0) = 0$ has a solution $x(t)$ such that $x(T^*) = 0$.

Stated more informally, on any such set of initial data, $S \cap P$, there is a point x^* and corresponding $T^* > 0$ such that if the string starts from rest at x^* , then at time T^* , all of the beads lie on ℓ . Duffin also observed that if $f_i(s)$ is odd in s , $1 \leq i \leq n$, (1.1) possesses a $4T^*$ periodic solution obtained from the above one by appropriate reflections about 0 and T^* . Duffin called the type of solution obtained in Theorem 1.2 an "oppositional mode of vibration". For linear f_i , this mode has the highest frequency.

During a recent visit to Carnegie-Mellon University, Duffin posed to us the question of whether there exist solutions of the type obtained above which start on S outside of P and after some time interval all lie on ℓ . In §3 we will give some partial answers to this question. In §2, some sharper versions of Theorem 1.2 will be obtained for a more general class of equations than (1.1). In particular we will prove that (1.1) possesses an unbounded connected set of initial data in P of the type given by Theorem 1.2.

§2. Continua of positive solutions

Let $x = (x_1, \dots, x_n)$ and let $g(x) = (g_1(x), \dots, g_n(x))$ satisfy

$$(g_1) \quad g \in C(\mathbb{R}^n, \mathbb{R}^n)$$

and

$$(g_2) \quad g: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}.$$

Note that the g defined by (1.1) and (1.3) satisfies (g_2) . Consider the system of ordinary differential equations:

$$(2.1) \quad m_i x_i'' + g_i(x) = 0 \quad 1 \leq i \leq n$$

where $m_i > 0$. In this section more general versions of Theorem 1.2 will be established for (2.1). By replacing $g_i(x)$ by $g_i(x)m_i^{-1}$, we can assume $m_i = 1$ and study

$$(2.2) \quad x'' + q(x) = 0.$$

We will seek solutions of (2.2) which pass through the origin at say $t = 0$ and at a second value of t , say $t = T$, have zero velocity, i.e. $x'(T) = 0$. If $x(t)$ is such a solution and $y(t) = x(T - t)$, then $y(0) = x(T)$, $y'(0) = x'(T) = 0$, $y(T) = x(0) = 0$, and $y''(t) = x''(T - t) = -g(x(T - t)) = -g(y(t))$. Therefore y is a solution of the type Duffin found. Our main result will be the existence of an unbounded connected set of data in P for which there exist solutions of Duffin type for (2.2). To state this more precisely, note that the time T at which $x' = 0$ is not known a priori. By rescaling the time variable, $t \rightarrow T^{-1}t$, we can make the new time interval 1. Still calling t the independent variable, (2.2) becomes

$$(2.3) \quad x'' + T^2 g(x) = 0$$

and

$$(2.4) \quad x(0) = 0 = x'(1)$$

Thus we seek $T \in (0, \infty)$ and x in the class of functions satisfying (2.4) such that (T, x) satisfies (2.3). Using (2.4) to integrate (2.3) shows

$$(2.5) \quad x'(t) = T^2 \int_t^1 g(x(\tau)) d\tau$$

and

$$(2.6) \quad x(t) = T^2 \int_0^t \left(\int_s^1 g(x(\tau)) d\tau \right) ds \equiv T^2 Kx$$

Solutions of (2.2) of the desired type will be obtained from the equivalent formulation (2.6).

Let E denote the set of $x \in C([0, 1], \mathbb{R}^n)$ satisfying (2.4). The form of K in (2.6) and (g_1) imply $K: E \rightarrow E$ and K is compact. Set

$$P = \{y \in E \mid y_i(t) > 0, \quad t \in [0, 1], \quad 1 \leq i \leq n\}.$$

Then P is a closed cone in E and (2.6) and (g_2) show $K: P \rightarrow P$. It is convenient to replace K by a new operator with a stronger positivity property. Let $\epsilon > 0$ and define $\underline{\epsilon} \in \mathbb{R}^n$ by $\underline{\epsilon} = (\epsilon, \dots, \epsilon)$. Now set

$$(2.7) \quad (K_\epsilon x)(t) \equiv \int_0^t \left(\int_s^1 (g(x(\tau)) + \underline{\epsilon}) d\tau \right) ds$$

Let

$$Q = \{y \in P \mid y_i'(0) > 0, \text{ and } y_i(t) > 0, t \in (0,1], 1 \leq i \leq n\}.$$

Then (2.7) shows $K_\epsilon: P \rightarrow Q$ and is compact.

Consider the operator equation

$$(2.8) \quad x = \mu K_\epsilon x$$

By a solution of (2.8) we mean a pair $(\mu, x) \in \mathbb{R} \times E$ satisfying (2.8). Our goal is to obtain information about solutions of (2.8) (with $\mu > 0$) for $\epsilon > 0$ and use it to deduce results about solutions of (2.6). There is a useful theorem about operator equations of the form (2.8) that can be exploited for this purpose. Let

$$\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}.$$

Proposition 2.9: Let E be a real Banach space and $K: E \rightarrow E$ be compact. Suppose there exists a closed cone P in E such that $K: P \rightarrow P$. Then the operator equation

$x = \mu K x$ has a component — i.e. a maximal closed connected set — of solutions, C , such that $(0,0) \in C$ and C is unbounded in $\mathbb{R}^+ \times P$.

Proof: Proposition 2.9 probably exists in the literature but we do not know a reference. Here is a brief proof. Consider K restricted to P . By the Dugundji Extension Theorem [2], there exists an extension \hat{K} of K to E which is compact and whose range lies in P . Consider the equation

$$(2.10) \quad x = \mu \hat{K} x$$

By Theorem 3.2 of [3], (2.10) has a component C of solutions meeting $(0,0)$ and unbounded in $\mathbb{R}^+ \times E$. Since the range of \hat{K} lies in P , $C \subset \mathbb{R}^+ \times P$. But in P , $\hat{K} = K$ so the proposition obtains.

Applying Proposition 2.9 to our K_ϵ and P and recalling that $K_\epsilon: P \rightarrow Q$, we see that (2.8) possesses a component of solutions, C_ϵ , meeting $(0,0)$ and unbounded in $\mathbb{R}^+ \times Q$. It is now an easy exercise to let $\epsilon \rightarrow 0$ and use C_ϵ and the compactness of K to pass to a limit and get a component of solutions C of (2.6) which is unbounded in $\mathbb{R} \times P$. However it may be the case that $q(0) = 0$ and if so, $(\mu, 0)$ is a trivial

solution of (2.6) for all $\mu \in \mathbb{R}^+$. Then C possibly just consists of these trivial solutions. We will show that even if $g(0) = 0$, (2.6) has lots of nontrivial solutions.

Towards that end, define a mapping of $\mathbb{R} \times E$ to \mathbb{R}^n by $P_1(\mu, x) = x(1)$. The form of (2.8) shows if $(\mu, x) \in C_\varepsilon$ and $\mu = 0$, then $x = 0$. Therefore $0 \in P_1 C_\varepsilon$. Since C_ε is a connected set and P_1 is continuous, $P_1 C_\varepsilon$ is a connected subset of \mathbb{R}^n which lies in P . Letting $\varepsilon \rightarrow 0$ will then yield:

Theorem 2.11: Let g satisfy (g_1) and (g_2) . Then there exists an unbounded connected set $D \subset P$ such that $0 \in D$ and for any $\eta \in D \setminus \{0\}$, there exists a solution x of (2.2) and $T(\eta) > 0$ such that $x(0) = 0$, $x(T) = \eta$, and $x'(T) = 0$.

Two preliminary results are needed to obtain Theorem 2.11.

Lemma 2.12: For each $\varepsilon > 0$, set $X_\varepsilon = \{x \in E \mid (\mu, x) \in C_\varepsilon\}$. Then X_ε is unbounded and connected.

Proof: By Proposition 2.9, and above remarks, C_ε is unbounded and connected in $\mathbb{R}^+ \times E$. Therefore X_ε , its projection on E is connected. If X_ε is bounded, there is a sequence of points $(\mu_m, x_m) \in C_\varepsilon$ such that $\mu_m \rightarrow \infty$ and x_m is bounded in E . The equation satisfied by (μ_m, x_m) and (g_2) yield

$$(2.13) \quad \frac{x_{m1}(t)}{m} = \int_0^t \left(\int_s^1 (g_1(x_m(\tau))^t \varepsilon) d\tau \right) ds > \int_0^t \varepsilon(1-s) ds = -\frac{\varepsilon}{2} (1-s)^2 \Big|_0^t$$

for each component x_{m1} of x_m . Setting $t = 1$ and letting $m \rightarrow \infty$ shows

$$(2.14) \quad 0 > \frac{\varepsilon}{2},$$

a contradiction. Therefore X_ε is unbounded in E .

Lemma 2.15: For each $\varepsilon > 0$, $P_1 C_\varepsilon$ is unbounded in P .

Proof: Equation (2.13) shows that if $(\mu, x) \in C_\varepsilon$, then $x_1^m(t) < 0$ in $(0, 1)$, $x_1^1(t) > 0$ in $(0, 1)$, and $x_i(t) > 0$ in $(0, 1)$ for $1 < i \leq m$. Therefore each component of x (resp. x') achieves its maximum in $[0, 1]$ at $t = 1$ (resp. $t = 0$). Consequently

$$(2.16) \quad |x(1)| = \max_{t \in [0, 1]} |x(t)| \equiv \|x\|_\infty$$

and by Lemma 2.12, $P_1 C_\varepsilon$ is unbounded and clearly lies in P .

Proof of Theorem 2.11: A standard theorem from point set topology [4, Chapter 1]

guarantees the existence of \bar{D} provided that for any neighborhood \bar{O} of 0 in $\mathbb{R}^n \cap P$, there is a $\xi \in \partial\bar{O}$, $\mu(\xi) > 0$, and a solution x of (2.3) - (2.4) with $T^2 = \mu$ and $x(1) = \xi$. Thus choose any such set \bar{O} and a decreasing sequence $\epsilon_m \rightarrow 0$. By Lemma 2.15 and previous remarks, for each m , $P_1 C_{\epsilon_m}$ is an unbounded connected subset of P which contains 0 . Hence there is a $(\mu_m, x_m) \in C_{\epsilon_m}$ with $x_m(1) \in \partial\bar{O}$. Since ∂ is bounded in P and the maximum of $|x_m(t)|$ occurs at $t = 1$, (x_m) is a bounded sequence in L^∞ . Suppose for the moment that the μ_m 's are also bounded. Then differentiating (2.13) shows that (μ_m, x_m) is bounded in $\mathbb{R}^+ \times C^2$. Consequently using (2.13) again we can assume (μ_m, x_m) converges in $\mathbb{R}^+ \times C^2$ to some (μ, x) , a solution of (2.6) with $x \in P$ and $x(1) \in \partial\bar{O}$. Clearly $\mu > 0$ for otherwise $x = 0$ contrary to $x(1) \in \partial\bar{O}$. Thus the theorem is proved once we establish that (μ_m) is a bounded sequence.

If (μ_m) is not bounded, we can assume $\mu_m \rightarrow \infty$ as $m \rightarrow \infty$. Consider $(x_m(1)) \subset \partial\bar{O}$. Therefore there is an $\alpha > 0$ and a subsequence of x_m such that for at least one component of x_m , say x_{mj} , we have $x_{mj}(1) > \alpha$ along the subsequence. Since $x_{mj}(t)$ is a concave function, $x_{mj}(t) > \frac{\alpha}{2}$ for all $t \in [\frac{1}{2}, 1]$. By the boundedness of \bar{O} again, there is a constant $M > 0$ such that $\|x_m\|_{L^\infty} \leq M$ independently of m . Hypothesis (g_2) implies there is a $\beta > 0$ such that

$$|g(\xi)|_{\mathcal{H}_1} \equiv \sum_{i=1}^n |g_i(\xi)| > \beta$$

for all ξ such that $\frac{\alpha}{2} \leq \xi_j \leq M$ and $0 \leq \xi_i \leq M$, $i \neq j$. Therefore by (2.13),

$$\begin{aligned}
 (2.17) \quad & \frac{|x_m(1)|_{L_1}}{\mu_m} > \int_0^1 \left(\int_s^1 |g(x_m(\tau))|_{L_1} d\tau \right) ds \\
 & > \int_0^{1/2} \left(\int_{1/2}^1 |g(x_m(\tau))|_{L_1} d\tau \right) ds > \frac{\beta}{4}
 \end{aligned}$$

Letting $m \rightarrow \infty$ in (2.17) leads to

$$(2.18) \quad 0 > \frac{\beta}{4},$$

a contradiction. Therefore μ_m is a bounded sequence and Theorem 2.11 is proved.

For each $\xi \in D \setminus \{0\}$, there is a $\mu > 0$ and $x \in P$ such that (μ, x) is a solution of (2.6) with $x(1) = \xi$. It is natural to ask whether there is a component, C , of solution of (2.6) in $R^+ \times P$ such that $P_1 C = D$. We do not believe such a simple statement holds without more assumptions. However in this direction we have

Corollary 2.19: For each $\xi \in D \setminus \{0\}$, there exists an unbounded connected set A_ξ of solutions of (2.6) in $R^+ \times P$ and a point $(\mu, x) \in A_\xi$ such that $P_1(\mu, x) = \xi$.

Proof: Let $\xi \in D$. By construction, $\xi = \lim \xi_m$ where $\xi_m = P_1(\mu_m, x_m)$ and

$(\mu_m, x_m) \in C_{\epsilon_m}$. Moreover (μ_m, x_m) converges in $R^+ \times C^2$ to $(\mu, x) \in R^+ \times P$ and

$P_1(\mu, x) = \xi$. Consider any bounded open set Ω in $R^+ \times P$ which contains (μ, x) . Then

$(\mu_m, x_m) \in \Omega$ for large m . Since C_{ϵ_m} is unbounded, $C_{\epsilon_m} \cap \partial\Omega \neq \emptyset$ for all large m .

It follows as in the proof of Theorem 2.11 that $\partial\Omega$ contains a solution of (2.6) and there exists A_ξ as stated above.

If $g(0) = 0$, (2.6) possesses the family of trivial solutions $\{(\mu, 0) | \mu \in R^+\}$ in $R^+ \times P$. If $g(0) \neq 0$, we have

Corollary 2.20: If $g(0) \neq 0$, (2.6) possesses an unbounded component of nontrivial solutions, C , in $R^+ \times P$ which meets $(0, 0)$ and satisfies $P_1 C$ is unbounded in P .

Proof: Let Ω be any bounded open neighborhood of $(0, 0)$ in $R^+ \times P$. The argument of

Corollary 2.19 or Theorem 2.11 shows that $\partial\Omega \cap C_{\epsilon_m} \neq \emptyset$ for all m . Hence as earlier there is an unbounded component C of solutions of (2.6) in $\mathbb{R}^+ \times P$ which meets $(0,0)$. Equation (2.6) shows if $(\mu, x) \in C$ and $\mu \neq 0$, then $x(t) > 0$ for $t > 0$ so $C \setminus \{(0,0)\}$ consists of nontrivial solutions. Lastly (2.17) shows $P_1 C$ cannot be bounded.

Remark 2.21: If S denotes the set of nontrivial solutions of (2.6) in $\mathbb{R}^+ \times P$, as we approach infinity along S , $x(1) \rightarrow 0$ or $x(1) \rightarrow \infty$, for otherwise the estimate (2.17) shows the corresponding μ 's must be bounded. Then (2.6) implies the x 's are also bounded, a contradiction.

To study the behavior of S more closely, in particular near $x = 0$ or ∞ requires more information about g near 0 and infinity. We will give a simple example of such an analysis.

By (2.6),

$$(2.22) \quad \frac{x(1)}{|x(1)|_{L_1}} = \mu \int_0^1 \left(\int_s^1 \frac{g(x(\tau))}{|x(1)|_{L_1}} d\tau \right) ds$$

Suppose that

$$(2.23) \quad |g(\xi)|_{L_1} > a|\xi|^\alpha$$

(Note that this holds for $\alpha = 1$ for the g obtained from (1.1) via (1.3)). Then (2.22)

- (2.23) show there is a constant a_1 such that

$$(2.24) \quad 1 > \mu a_1 |x(1)|_{L_1}^{\alpha-1}$$

If $\alpha > 1$, $\mu \rightarrow 0$ as $x(1) \rightarrow \infty$ while if $\alpha = 1$, $\mu < a_1^{-1}$. Since (2.24) also holds for (2.8) uniformly in $\epsilon > 0$, we see if $\alpha = 1$, C_ϵ and therefore any set obtained from it

by taking limits lies in $[0, a_1^{-1}] \times P$ and contains $(0,0)$.

In [1], Duffin conjectured that if S is as in Theorem 1.2, then there exists $\eta \in S$, $T(\eta) > 0$, and a solution $x(t)$ of (1.1) such that $x(0) = 0$, $x'(0) = \eta$,

$x'(T) = 0$, and $x(t) \in P$ for all $t \in [0, T]$. One cannot expect this to occur in the generality of (2.6). For example consider

$$(2.25) \quad x'' + V'(x) = 0$$

where $V \in C^2(P, \mathbb{R})$ and $g = V'$ satisfies (g_2) . Suppose further that

$V(x) \rightarrow 0$ uniformly as $|x| \rightarrow \infty$. (E.g. take $n = 1$ and $V(x) = -(x+1)^{-1}$). Since

(2.25) is a Hamiltonian system

$$(2.26) \quad \frac{1}{2} |x'(t)|^2 + V(x(t)) \equiv \text{constant}$$

independent of t for any solution x of (2.25). Let $T = \{\eta \in P \mid \text{there is } T > 0$

and x , a solution of (2.1) with $x(0) = 0$, $x'(0) = \eta$, $x'(t) = 0$, and

$x(t) \in P$ for all $t \in [0, T]\}$. We claim T is bounded in P . Indeed let $\eta \in T$ with corresponding $x(t)$. By (2.26),

$$(2.27) \quad \frac{1}{2} |x'(0)|^2 + V(x(0)) = \frac{1}{2} \eta^2 + V(0) = V(x(T))$$

If T were unbounded, (2.27) shows $V(P)$ must also be unbounded, contrary to our assumption on V at ∞ .

Next we will show that Duffin's conjecture holds in his setting.

Theorem 2.28: Suppose g satisfies (g_1) , (g_2) and (2.23) with $\alpha = 1$. Then T contains an unbounded component containing 0.

Proof. These hypotheses, Theorem 2.11, and the remarks following (2.21) show (2.6)

possesses a component of solutions, C , unbounded in $[0, a_1^{-1}] \times P$ with $(0, 0) \in C$. For

any $(\mu, x) \in C$, the change of time scale $t \rightarrow \tau = \mu^{1/2} t$, $x(t) = y(\tau) = x(\mu^{-1/2} \tau)$ transforms solutions of (2.3) - (2.4) to solutions of (2.2) with $y(0) = 0 = y'(\mu^{1/2})$.

Consider

$$\hat{T} = \{y'(0) = \mu^{-1/2} x'(0) \mid (\mu, x) \in C\}$$

A priori, $y'(0)$ is not defined for $(0, 0) \in C$. But if we approach $(0, 0)$ along C , by

(2.5)

$$\mu^{-1/2} x'(0) = \mu^{1/2} \int_0^1 g(x(\tau)) d\tau \rightarrow 0.$$

so $0 \in \hat{T}$. It follows that the map $(\mu, x) \mapsto \mu^{-1/2} x'(0)$, $C \times \hat{T}$ is continuous. Therefore \hat{T} is connected. Since $(\mu, x) \in C$ implies $\mu \in [0, a_1^{-1}]$ while $\{x'(0)\}$ is unbounded via (2.5) and the unboundedness of $\{x(0)\}$ in E , \hat{T} must be unbounded and the Theorem is proved.

§3. Some remarks on other solutions of (2.2)

Another question Duffin posed in [1] is whether (1.1) possesses solutions of the type found earlier other than "positive" ones. Some partial answers will be given in this section.

To begin suppose g satisfies

$$(g_4) \quad g(\xi) = M\xi + o(|\xi|) \text{ as } \xi \rightarrow 0$$

where M is an $n \times n$ matrix and $M: P \rightarrow P$. Associated with (2.3) - (2.4) is a linear eigenvalue problem

$$(3.1) \quad y'' + \rho^2 My = 0, \quad 0 < t < 1$$

$$(3.2) \quad y(0) = 0 = y'(1)$$

An eigenfunction of (3.1) - (3.2) must have the form $y(t) = (\sin(k - \frac{1}{2})\pi t)c$ where $k \in \mathbb{N}$, $c \in \mathbb{R}^n$, and satisfies

$$(3.3) \quad (k - \frac{1}{2})^2 \pi^2 c = \rho^2 Mc.$$

If ρ^2 is a simple (or odd multiplicity) eigenvalue of (3.1) - (3.2), a global bifurcation theorem from [3] provides the existence of nontrivial solutions of (2.2) bifurcating from $(\rho^2, 0)$.

Rather than work in this generality, we specialize to Duffin's case where

$f_j(s) = -a_j s + o(s)$ as $s \rightarrow 0$ in \mathbb{R} , $1 \leq j \leq n+1$ where $a_j > h > 0$. Then (3.1) becomes

$$(3.4) \quad y_i'' = -\frac{a_i}{m_i} (y_{i-1} + y_i) - \frac{a_{i+1}}{m_i} (y_i + y_{i+1}),$$

$1 \leq i \leq n$ and $y_0 = 0 = y_{n+1}$. Since $m_i, a_j > 0$, M is a tridiagonal matrix with positive entries along its main and two neighboring diagonals. A computation shows that the determinants of the principle minors of M are all positive. Consequently M is an

$x'(T) = 0$, and $x(t) \in P$ for all $t \in [0, T]$. One cannot expect this to occur in the generality of (2.6). For example consider

$$(2.25) \quad x'' + V'(x) = 0$$

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(2.25) is a Hamiltonian system

$$(2.26) \quad \frac{1}{2} |x'(t)|^2 + V(x(t)) \equiv \text{constant}$$

independent of t for any solution x of (2.25). Let $T = \{\eta \in P \mid \text{there is } T > 0 \text{ and } x, \text{ a solution of (2.1) with } x(0) = 0, x'(0) = \eta, x'(t) = 0, \text{ and } x(t) \in P \text{ for all } t \in [0, T]\}$. We claim T is bounded in P . Indeed let $\eta \in T$ with corresponding $x(t)$. By (2.26),

$$(2.27) \quad \frac{1}{2} |x'(0)|^2 + V(x(0)) = \frac{1}{2} \eta^2 + V(0) = V(x(T))$$

If T were unbounded, (2.27) shows $V(P)$ must also be unbounded, contrary to our assumption on V at ∞ .

Next we will show that Duffin's conjecture holds in his setting.

Theorem 2.28: Suppose g satisfies (g_1) , (g_2) and (2.23) with $\alpha = 1$. Then T contains an unbounded component containing 0.

Proof. These hypotheses, Theorem 2.11, and the remarks following (2.21) show (2.6) possesses a component of solutions, C , unbounded in $[0, a_1^{-1}] \times P$ with $(0, 0) \in C$. For any $(\mu, x) \in C$, the change of time scale $t \rightarrow \tau = \mu^{1/2} t$, $x(t) = y(\tau) = x(\mu^{-1/2} \tau)$ transforms solutions of (2.3) - (2.4) to solutions of (2.2) with $y(0) = 0 = y'(\mu^{1/2})$.

Consider

$$\hat{T} = \{y'(0) = \mu^{-1/2} x'(0) \mid (\mu, x) \in C\}$$

A priori, $y'(0)$ is not defined for $(0, 0) \in C$. But if we approach $(0, 0)$ along C , by

$$(2.5) \quad \mu^{-1/2} x'(0) = \mu^{1/2} \int_0^1 g(x(\tau)) d\tau \rightarrow 0.$$

so $0 \in \hat{T}$. It follows that the map $(\mu, x) \mapsto \mu^{-1/2} x'(0)$, $C \rightarrow \hat{T}$ is continuous. Therefore \hat{T} is connected. Since $(\mu, x) \in C$ implies $\mu \in [0, a_1^{-1}]$ while $\{x'(0)\}$ is unbounded via (2.5) and the unboundedness of $\{x(0)\}$ in E , \hat{T} must be unbounded and the Theorem is proved.

§3. Some remarks on other solutions of (2.2)

Another question Duffin posed in [1] is whether (1.1) possesses solutions of the type found earlier other than "positive" ones. Some partial answers will be given in this section.

To begin suppose g satisfies

$$(g_4) \quad g(\xi) = M\xi + o(|\xi|) \text{ as } \xi \rightarrow 0$$

where M is an $n \times n$ matrix and $M: P \rightarrow P$. Associated with (2.3) - (2.4) is a linear eigenvalue problem

$$(3.1) \quad y'' + \rho^2 M y = 0, \quad 0 < t < 1$$

$$(3.2) \quad y(0) = 0 = y'(1)$$

An eigenfunction of (3.1) - (3.2) must have the form $y(t) = (\sin(k - \frac{1}{2})\pi t)c$ where $k \in \mathbb{N}$, $c \in \mathbb{R}^n$, and satisfies

$$(3.3) \quad (k - \frac{1}{2})^2 \pi^2 c = \rho^2 M c.$$

If ρ^2 is a simple (or odd multiplicity) eigenvalue of (3.1) - (3.2), a global bifurcation theorem from [3] provides the existence of nontrivial solutions of (2.2) bifurcating from $(\rho^2, 0)$.

Rather than work in this generality, we specialize to Duffin's case where

$f_j(s) = -a_j s + o(s)$ as $s \rightarrow 0$ in \mathbb{R} , $1 \leq j \leq n+1$ where $a_j > h > 0$. Then (3.1) becomes

$$(3.4) \quad y_i'' = -\frac{a_i}{m_i} (y_{i-1} + y_i) - \frac{a_{i+1}}{m_i} (y_i + y_{i+1}),$$

$1 \leq i \leq n$ and $y_0 = 0 = y_{n+1}$. Since $m_i, a_j > 0$, M is a tridiagonal matrix with positive entries along its main and two neighboring diagonals. A computation shows that the determinants of the principle minors of M are all positive. Consequently M is an

oscillation matrix [5] and its eigenvalues are all positive and simple:

$v_1 > \dots > v_n > 0$ and any eigenvector c_j corresponding to v_j has $j - 1$ sign changes. (This means if $c_j = (c_{j1}, \dots, c_{jn})$, the sequence of components changes sign exactly $j-1$ times). In particular c_1 has no sign changes and we can assume $c_1 \in P$.

The eigenvalues of (3.1) - (3.2) are then

$$(3.5) \quad \left\{ \frac{(k - \frac{1}{2})^2 \pi^2}{v_i} \mid 1 \leq i \leq s_1, k \in \mathbb{N} \right\}$$

These eigenvalues may not be simple; possibly $(k - \frac{1}{2})^2 v_i^{-1} = (j - \frac{1}{2})^2 v_p^{-1}$ for some choice of indices. However for most choices of a_i, m_j they will be simple. In any event we have

Theorem 3.6: Let $f_i \in C^1$ and satisfy (1.3), $1 \leq i \leq n+1$. Suppose ρ^2 is an eigenvalue of (3.1) - (3.2) of odd multiplicity. Then (1.1) possesses a component \hat{C} of nontrivial solutions in $\mathbb{R}^+ \times E$ which contains $(\rho^2, 0)$ and either is unbounded or meets $(r^2, 0)$ for some other eigenvalue r^2 of (3.1) - (3.2).

Proof. The system (1.1) is eigenvalent to the compact operator equation (2.6) where $Kx = Lx + o(x)$ as $x \rightarrow 0$ and

$$(Lx)(t) = \int_0^t \left(\int_s^1 Mx(\tau) d\tau \right) ds.$$

Since ρ^2 is of odd multiplicity the conclusion follows immediately from the global bifurcation theorem of [3].

Remark 3.7: Since $v_j > v_1$ for $j > 1$, (3.5) shows $\mu_1 \equiv (\mu v_1)^{-1} \pi^2$ is the smallest eigenvalue of (3.1) - (3.2) and is simple. A standard bifurcation theorem then says that near $(\mu_1, 0)$, \hat{C} is a curve of the form $\mu(s) = \mu_1 + o(1)$, $x(s) = sy_1 + o(|s|)$ for s near 0. Since $c_1 \in P$, $y_1 = (\sin \frac{\pi t}{2}) c_1 \in P$. In fact $y_1 \in Q$ which is an open set in the C^1 topology. Then for $s > 0$ and small, $(\mu(s), x(s))$ lies in $\mathbb{R}^+ \times Q$. Under a stronger assumption on g as defined by (1.1), this curve extends to an unbounded connected set of solutions of (1.1) in $\mathbb{R}^+ \times Q$.

Corollary 3.8: Under the hypotheses of Theorem 3.6, suppose $g: P \setminus \{0\}$ into the interior of P . Then (1.1) contains a component of solutions C^+ unbounded in $R^+ \times Q$ and which meets $(\mu_1, 0)$.

Proof: By Remark 3.7 and Theorem 3.6, we need only show that $\hat{C} \cap (R^+ \times Q)$ has the desired property. Results from [3] or [7] show that the curve mentioned in Remark 3.7 either is part of an unbounded connected set of solutions of (1.1) in $R^+ \times Q$ or it meets a point $(\mu, x) \in R^+ \times \partial Q$ other than $(\mu_1, 0)$. If the latter case occurs, there is a sequence $(\mu_m, x_m) \in \hat{C} \cap (R^+ \times Q)$ such that $(\mu_m, x_m) \rightarrow (\mu, x)$. This implies some component z of x satisfies $z'(0) = 0$ or $z(t) = 0$ for some $t \in [0, 1]$. But (2.5) - (2.6) and our additional assumption on g show this is only possible if $z = 0$ and $x = 0$. Then $(\mu_m, x_m) \rightarrow (\mu, 0)$ where μ lies in the set defined by (3.5). Writing (2.6) as

$$(3.9) \quad \frac{x_m}{\|x_m\|} = \mu_m \frac{Kx_m}{\|x_m\|}$$

and letting $m \rightarrow \infty$, we see $x_m \|x_m\|^{-1}$ converges to a solution of

$$(3.10) \quad y = \mu Ly$$

where $y \in P$, $\|y\| = 1$, $\mu \neq \mu_1$. We claim this is impossible. Indeed y has the form

$$y = \sum \alpha_p \left(\sin(k_p - \frac{1}{2})\pi t \right) c_{1_p}$$

where the α_p 's are scalars, each c_j appears at most once, and if $c_{1_p} = c_1$, $k_p > 1$.

Let c_1^* denote an eigenvalue of M^* , the adjoint M corresponding to v_1 . We can assume $c_1^* \in P$. Let $y_1^* = (\sin \frac{\pi}{2} t) c_1^*$. Since c_1^* is orthogonal to c_k if $k \neq 1$, taking the inner product of (3.9) with y_1^* yields

$$(3.11) \quad \int_0^1 y \cdot y_1^*, dt = 0$$

But $y_1^* \in Q$ and $\|y\| = 1$ implies the left hand side of (3.11) must be positive. Thus $\hat{C} \cap (R^+ \times Q)$ is unbounded and the proof is complete.

Remark 2.12: As was noted earlier, the eigenvalues (3.5) of (3.1) - (3.2) are simple in general and then the argument of Remark 3.7 and [3] or [7] give curves and continua of

solutions of (1.1) emanating from the corresponding bifurcation points. Globally the argument of Corollary 3.8 cannot be used to keep these solution branches from meeting each other. However we see that in general there are many distinct small amplitude solutions of (1.1) which do not lie in $\mathbb{R}^+ \times P$. In particular the small amplitude solutions associated with the eigenvalues $(4\nu_j)^{-1}\pi^2$ will have the same "nodal" shapes as the eigenvectors c_j , $1 \leq j \leq n$.

Variational methods can also be used to obtain information about small solutions of (1.1) irrespective of multiplicity considerations for the eigenvalues of (3.1) - (3.2). A brief sketch of how this can be done will be given. Let

$$F_i(s) = \int_0^s f_i(z) dz, \quad 1 \leq i \leq n+1$$

and let

$$g_i(x) = -f_i(x_{i-1} + x_i) - f_i(x_i + x_{i+1})$$

for $1 \leq i \leq n$ where $x_0 = x_{n+1} = 0$. Note that if

$$V(x) = - \sum_{i=1}^{n+1} F_i(x_{i-1} + x_i),$$

then $\frac{\partial V}{\partial x_i}(x) = g_i(x)$. Consider the variational problem: Find critical points of

$$(3.13) \quad I(x) = \int_0^2 V(x(t)) dt$$

subject to the constraint

$$(3.14) \quad \frac{1}{2} \int_0^2 \sum_{i=1}^n m_i (x_i'(t))^2 dt = r^2$$

where x lies in the class of functions

$$\{x(t) \in W^{1,2}([0,2], \mathbb{R}^n) \mid x(0) = 0 \text{ and } x(1+s) = x(1-s), s \in [0,1]\}$$

The left hand side of (3.15) can be taken as the square of the norm in this space. The Euler equations satisfied by any smooth critical point of this problem are

$$(3.16) \quad m_i x_i'' + \mu g_i(x) = 0, \quad 1 \leq i \leq n$$

i.e. (2.3) where μ^{-1} is the corresponding Lagrange multiplier. The choice of function space shows x also satisfies (2.4).

A bifurcation theorem due to Böhme [8] or Marino [9] can be applied to the above situation and shows that each eigenvalue of (3.1) - (3.2) is a bifurcation point for (1.1). However it does not give us information, for a multiple eigenvalue, about the shape of the corresponding solution of (1.1).

Remark 3.17: Results of van Groesen [10] (based on work of Ekeland and Lasry [11] show that if

$$H(x, \dot{x}) = \frac{1}{2} \sum m_i |\dot{x}_i|^2 + V(x)$$

and V is even, convex, and satisfies other technical geometrical conditions, then for each $c > 0$, the energy surface $H = c$ contains at least n distinct solutions of (1.1) of the type we seek. No information is obtained however about their shape.

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